RECOGNISING ACHIEVEMENT

ADVANCED GCE **4758/01**

MATHEMATICS (MEI)

Differential Equations

THURSDAY 12 JUNE 2008 Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):

Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

1 Fig. 1 shows a particle of mass 2 kg suspended from a light vertical spring. At time t seconds its displacement is x m below its equilibrium level and its velocity is $v\,\text{m s}^{-1}$ vertically downwards. The forces on the particle are

- its weight, $2g$ N
- the tension in the spring, $8(x + 0.25g)$ N
- the resistance to motion, $2kv$ N where k is a positive constant.

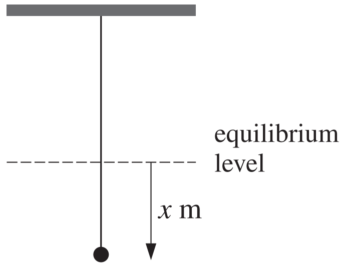


Fig. 1

(i) Use Newton's second law to write down the equation of motion for the particle, justifying the signs of the terms. Hence show that the displacement is described by the differential equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 4x = 0.$$ [4]

The particle is initially at rest with $x = 0.1$.

(ii) In the case $k = 0$, state the general solution of the differential equation. Find the solution, subject to the given initial conditions. [4]

(iii) In the case $k = 2$, find the solution of the differential equation, subject to the given initial conditions. Sketch a graph of the solution for $t \geqslant 0$. [11]

(iv) Find the range of values of k for which the system is over-damped. Sketch a possible graph of the solution in such a case. [5]

2 The radioactive substance X decays into the substance Y, which in turn decays into Z. At time t hours the masses, in grams, of X, Y and Z are denoted by x, y and z respectively.

Initially there is 8 g of X and there is no Y or Z present.

The differential equation modelling the decay of X is $\frac{dx}{dt} = -2x$.

(i) Find x in terms of t. [3]

The differential equation modelling the amount of Y is $\frac{dy}{dt} = 2x - y$.

(ii) Using your expression for x found in part **(i)**, solve this equation to find y in terms of t. [9]

(iii) Show that $y > 0$ for $t > 0$. Sketch a graph of y for $t \geqslant 0$. [5]

The differential equation modelling the amount of Z is $\frac{dz}{dt} = y$.

(iv) Without solving this equation, show that $x + y + z = 8$. Hence show that $z = 8\left(1 - e^{-t}\right)^2$. [5]

(v) Calculate the time required for 99% of the total mass to become substance Z. [2]

3 The differential equation $t\frac{\mathrm{d}y}{\mathrm{d}t} + ky = t$, where k is a constant, is to be solved for $t \geqslant 1$, subject to the condition $y = 0$ when $t = 1$.

(i) When $k \neq -1$, find the solution for y in terms of t and k. [10]

(ii) Sketch a graph of the solution for $k = 2$. [2]

(iii) When $k = -1$, find the solution for y in terms of t. [5]

Now consider the differential equation $t\frac{\mathrm{d}y}{\mathrm{d}t} - \sin y = t$, subject to the condition $y = 0$ when $t = 1$. This is to be solved by Euler's method. The algorithm is given by $t_{r+1} = t_r + h$, $y_{r+1} = y_r + h\dot{y}_r$.

(iv) Using a step length of 0.1, perform two iterations of the algorithm to estimate the value of y when $t = 1.2$. [4]

If the algorithm is carried out with a step length of 0.05, the estimate for y when $t = 1.2$ is $y \approx 0.2138$.

(v) Explain with a reason which of these two estimates for y when $t = 1.2$ is likely to be more accurate. Hence, or otherwise, explain whether these estimates are likely to be overestimates or underestimates. [3]

4 The simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - 6y - 9\sin t,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3x - 5y - 7\sin t,$$

are to be solved.

(i) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} - 2x = -9\cos t - 3\sin t$. [6]

(ii) Find the general solution for x. [9]

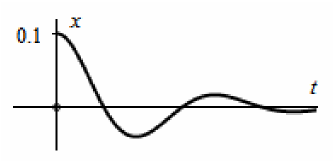
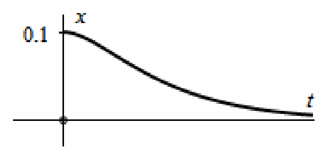
(iii) Hence find the corresponding general solution for y. [3]

It is given that x is bounded as $t \to \infty$.

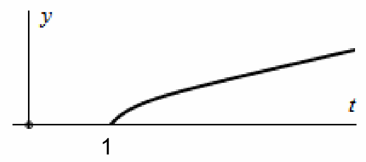
(iv) Show that y is also bounded as $t \to \infty$. [2]

(v) Given also that $y = 0$ when $t = 0$, find the particular solutions for x and y. Write down the expressions for x and y as $t \to \infty$. [4]

4758 Differential Equations

1 (i)	$2\ddot{x} = 2g - 8(x + 0.25g) - 2kv$	M1	N2L equation with all forces using given expressions for tension and resistance	
	Weight positive as down, tension negative as up.	B1		
	Resistance negative as opposes motion.	B1		
	$\Rightarrow \ddot{x} + k\dot{x} + 4x = 0$	E1	Must follow correct N2L equation	4
(ii)	$x = A\cos 2t + B\sin 2t$	B1		
	$t = 0, x = 0.1 \Rightarrow A = 0.1$	M1	Find the coefficient of cos	
	$\dot{x} = -2A\sin 2t + 2B\cos 2t$ so $t = 0, \dot{x} = 0 \Rightarrow B = 0$	M1	Find the coefficient of sin	
	$x = 0.1\cos 2t$	A1	cao	4
(iii)	$\alpha^2 + 2\alpha + 4 = 0$	M1	Auxiliary equation	
	$\alpha = -1 \pm \sqrt{3}\,j$	A1		
		M1	CF for complex roots	
	$x = e^{-t}\left(C\cos\sqrt{3}\,t + D\sin\sqrt{3}\,t\right)$	F1	CF for their roots	
	$t = 0, x = 0.1 \Rightarrow C = 0.1$	M1	Condition on x	
	$\dot{x} = -e^{-t}\left(C\cos\sqrt{3}\,t + D\sin\sqrt{3}\,t\right)$ $+e^{-t}\left(-\sqrt{3}\,C\sin\sqrt{3}\,t + \sqrt{3}\,D\cos\sqrt{3}\,t\right)$	M1	Differentiate (product rule)	
	$0 = -C + \sqrt{3}\,D$	M1	Condition on $\dot{x}$	
	$D = \frac{0.1}{\sqrt{3}}$			
	$x = 0.1e^{-t}\left(\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t\right)$	A1	cao	
		B1 B1 B1	Curve through (0,0.1) with zero gradient Oscillating Asymptote x = 0	11
(iv)	$k^2 - 4\cdot 1\cdot 4 > 0$	M1	Use of discriminant	
		A1	Correct inequality	
	(As *k* is positive) $k > 4$	A1	Accept $k < -4$ in addition (but not *k* > –4)	
		B1 B1	Curve through (0,0.1) Decays without oscillating (at most one intercept with positive *t* axis)	5

2 (i)	$x = A\mathrm{e}^{-2t}$	M1	Any valid method	
	$t = 0, x = 8 \Rightarrow A = 8$	M1	Condition on x	
	$x = 8\mathrm{e}^{-2t}$	A1		
				3
(ii)	$\dot{y} + y = 16\mathrm{e}^{-2t}$	M1	Substitute for x	
	$\alpha + 1 = 0 \Rightarrow \alpha = -1$	M1	Auxiliary equation	
	CF $y = B\mathrm{e}^{-t}$	A1		
	PI $y = a\mathrm{e}^{-2t}$	B1		
	$-2a\mathrm{e}^{-2t} + a\mathrm{e}^{-2t} = 16\mathrm{e}^{-2t}$	M1	Differentiate and substitute	
	$a = -16$	A1	cao	
	GS $y = -16\mathrm{e}^{-2t} + B\mathrm{e}^{-t}$	F1	Their PI + CF (with one arbitrary constant)	
	$t = 0, y = 0 \Rightarrow B = 16$	M1	Condition on y	
	$y = 16\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right)$	F1	Follow a non-trivial GS	
	Alternative mark scheme for first 7 marks:			
		M1	*Substitute for x*	
	$I = e^{t}$	*M1*	*Attempt integrating factor*	
		A1	*IF correct*	
	$d(y\,e^{t})/dt = 16e^{-t}$	*B1*		
		M1	*Integrate*	
	$y\,e^{t} = -16e^{-t} + B$	*A1*	*cao*	
	$y = -16e^{-2t} + Be^{-t}$	*F1*	*Divide by their I (must divide constant)*	
				9
(iii)	$y = 16\mathrm{e}^{-t}\left(1 - \mathrm{e}^{-t}\right)$	M1	Or equivalent (NB $e^{-t} > e^{-2t}$ needs justifying)	
	$16\mathrm{e}^{-t} > 0$ and $t > 0 \Rightarrow \mathrm{e}^{-t} < 1$ hence $y > 0$	E1	Complete argument	
	 	B1	Starts at origin	
		B1	General shape consistent with their solution and $y > 0$	
		B1	Tends to zero	
				5
(iv)	$\frac{\mathrm{d}}{\mathrm{d}t}(x + y + z) = (-2x) + (2x - y) + (y) = 0$	M1	Consider sum of DE's	
	$\Rightarrow x + y + z = c$	E1		
	Hence initial conditions $\Rightarrow x + y + z = 8$	E1		
	$z = 8 - x - y$	M1	Substitute for x and y and find z	
	$z = 8\left(1 - 2\mathrm{e}^{-t} + \mathrm{e}^{-2t}\right) = 8\left(1 - \mathrm{e}^{-t}\right)^2$	E1	Convincingly shown (x, y must be correct)	
				5
(v)	$0.99 \times 8 = 8\left(1 - \mathrm{e}^{-t}\right)^2$	B1	Correct equation (any form)	
	$t = -0.690638$ or 5.29581			
	99% is Z after 5.30 hours	B1	Accept value in [5.29, 5.3]	
				2

3 (i)	$\dot{y}+\frac{k}{t}y=1$	M1	Divide by *t* (condone LHS only)	
	$I=\exp\left(\int \frac{k}{t}\,\mathrm{d}t\right)=\exp(k\ln t)=t^k$	M1	Attempt integrating factor	
		A1	Integrating factor	
	$t^k\dot{y}+kt^{k-1}y=t^k$	F1	Multiply DE by their *I*	
	$\frac{\mathrm{d}}{\mathrm{d}t}\left(yt^k\right)=t^k$	M1	LHS	
	$yt^k=\int t^k\,\mathrm{d}t$	M1	Integrate	
	$=\frac{1}{k+1}t^{k+1}+A$	A1	cao (including constant)	
	$y=\frac{1}{k+1}t+At^{-k}$	F1	Divide by their *I* (must divide constant)	
	$t=1, y=0 \Rightarrow 0=\frac{1}{k+1}+A \Rightarrow A=-\frac{1}{k+1}$	M1	Use condition	
	$y=\frac{1}{k+1}\left(t-t^{-k}\right)$	F1	Follow a non-trivial GS	
				10
(ii)	$y=\frac{1}{3}\left(t-t^{-2}\right)$ 	B1 B1 B1	Shape consistent with their solution for $t \geq 1$ Passes through (1, 0) Behaviour for large *t*	
				3
(iii)	$yt^{-1}=\int t^{-1}\,\mathrm{d}t$	M1	Follow their (i)	
	$=\ln t+B$	A1	cao	
	$y=t(\ln t+B)$	F1	Divide by their *I* (must divide constant)	
	$t=1, y=0 \Rightarrow B=0 \Rightarrow y=t\ln t$	A1	cao	
				4
(iv)	$\frac{\mathrm{d}y}{\mathrm{d}t}=1+t^{-1}\sin y$	M1	Rearrange DE (may be implied)	
	t *y* d*y*/d*t* 1 0 1 1.1 0.1 1.0908 1.2 0.2091	M1 A1 A1	Use algorithm *y*(1.1) *y*(1.2)	
				4
(v)	0.2138 as smaller step size	B1	Must give reason	
	Decreasing step length has increased estimate. Assuming this estimate is more accurate, decreasing step length further will increase estimate further, so true value likely to be greater.	M1	Identify effect of decreasing step length	
	Hence underestimates.	A1	Convincing argument	
	Alternative mark scheme for last 2 marks:			
	dy/dt seems to be increasing, hence Euler's method	*M1*	*Identify derivative increasing*	
	will underestimate true value + sketch (or explanation).	*A1*	*Convincing argument*	
				3

4 (i)	$\ddot{x} = 4\dot{x} - 6\dot{y} - 9\cos t$	M1	Differentiate first equation	
	$= 4\dot{x} - 6(3x - 5y - 7\sin t) - 9\cos t$	M1	Substitute for $\dot{y}$	
	$y = \frac{1}{6}(4x - \dot{x} - 9\sin t)$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = 4\dot{x} - 18x + 5(4x - \dot{x} - 9\sin t) + 42\sin t - 9\cos t$	M1	Substitute for y	
	$\ddot{x} + \dot{x} - 2x = -3\sin t - 9\cos t$	E1	LHS	
		E1	RHS	6
(ii)	$\alpha^2 + \alpha - 2 = 0$	M1	Auxiliary equation	
	$\alpha = 1$ or -2	A1		
	CF $x = A\mathrm{e}^{t} + B\mathrm{e}^{-2t}$	F1	CF for their roots	
	PI $x = a\cos t + b\sin t$	B1	PI of this form	
	$(-ac - bs) + (-as + bc) - 2(ac + bs) = -3s - 9c$	M1	Differentiate twice and substitute	
	$-a + b - 2a = -9$	M1	Compare coefficients (2 equations)	
	$-b - a - 2b = -3$	M1	Solve (2 equations)	
	$\Rightarrow a = 3, b = 0$	A1		
	$x = 3\cos t + A\mathrm{e}^{t} + B\mathrm{e}^{-2t}$	F1	Their PI + CF (with two arbitrary constants)	9
(iii)	$y = \frac{1}{6}(4x - \dot{x} - 9\sin t)$	M1	y in terms of $x, \dot{x}$	
	$= \frac{1}{6}\left(12\cos t + 4A\mathrm{e}^{t} + 4B\mathrm{e}^{-2t} + 3\sin t - A\mathrm{e}^{t} + 2B\mathrm{e}^{-2t} - 9\sin t\right)$	M1	Differentiate x and substitute	
	$y = 2\cos t - \sin t + \frac{1}{2}A\mathrm{e}^{t} + B\mathrm{e}^{-2t}$	A1	Constants must correspond with those in x	3
(iv)	x bounded $\Rightarrow A = 0$	M1	Identify coefficient of exponentially growing term must be zero	
	$\Rightarrow y$ bounded	E1	Complete argument	2
(v)	$t = 0, y = 0 \Rightarrow 0 = B + 2 \Rightarrow B = -2$	M1	Condition on y	
	$x = 3\cos t - 2\mathrm{e}^{-2t}$, $y = 2\cos t - \sin t - 2\mathrm{e}^{-2t}$	F1	Follow their (non-trivial) general solutions	
	$x = 3\cos t$	A1	cao	
	$y = 2\cos t - \sin t$	A1	cao	4

4758 Differential Equations (Written Examination)

General Comments

Many candidates demonstrated a good understanding of the specification and high levels of algebraic competency, but some candidates struggled to apply the correct methods.

Candidates should note that if a general solution only is required, then this will be specified. When conditions have been given and a solution is asked for (without the word 'general' preceding it), then the conditions should be used to find the particular solution.

Candidates are again reminded that sketch graphs should show the initial or boundary conditions. Although follow through marks are often awarded when the solution to the differential equation is wrong, if the graph is not consistent with known information, marks will usually be lost. Detailed graph-sketching skills are not required, but sketches are often assessed on a 'beginning, middle and end' approach – i.e. do they 'begin in the right place' (showing the conditions) do they 'end in the right place' (e.g. behaviour for large values of the independent variable) and is their shape 'in the middle' approximately correct.

Comments on Individual Questions

1) (i) Many candidates were able to write down a correct equation of motion, although some omitted the weight. However, few candidates scored full marks as most either omitted to justify the signs (as requested) or gave incorrect reasons for the sign of the resistance force.

(ii) This was often done very well, although a few candidates made errors in their general solution.

(iii) Finding the general solution was often done very well. The sketch graph often omitted important details, in particular the initial velocity. Some graphs did not make it sufficiently clear that the curve was oscillating. Some did not make it clear that the graph tends to zero.

(iv) Most candidates knew to use the discriminant to determine the type of damping, but some candidates used the wrong sign. Sketch graphs were often good, but some were oscillating. A small minority of candidates seemed not to know what was required here.

2) (i) This was often done very well, but some candidates omitted to use the condition on x.

(ii) This was also often done well, some using the complementary function and particular integral method, others using the integrating factor method. Only a small minority incorrectly tried to separate variables.

(iii) Most candidates gave insufficient working to justify that y was positive. However sketch graphs were generally correct.

(iv) Many candidates either omitted to show that $x + y + z = 8$, or tried to solve the differential equation, despite the instruction not to solve it. Many argued from their knowledge of physics about conservation of mass. Only a minority took the correct approach of summing the differential equations to show that $x + y + z$ must be constant, and then using the initial conditions to find the value of the constant.

(v) Many correct solutions for the time were seen, although some made the solution harder by multiplying out the given expression which had been deliberately given in factorised form to help candidates.

3) (i) This was often done extremely well. Only a small number of candidates did not recognise the need to use the integrating factor method. A surprising minority did not use the laws of indices correctly – in particular dividing t^{k+1} by t^k to obtain t^k was a common error.

(ii) Sketch graphs often lacked the given condition being marked and the behaviour for large *t* being clear, but many good sketches were seen.

(iii) This was often completely correct, but some incorrectly used their answer to (i).

(iv) The numerical solution was often done well, although a sizeable minority made numerical errors in the second step.

(v) Many candidates knew that reducing the step size is likely to increase accuracy. Reasons whether the estimates were over- or underestimates were often unclear, lacking in evidence or omitted. Some gave very good explanations.

4) (i) Most candidates completed this correctly. Some made algebraic errors, but only very few did not know how to do the elimination.

(ii) This was often correct.

(iii) Many candidates correctly used their solution for x in the first of the displayed differential equations. It was good to see that very few candidates attempted to construct a differential equation for y.

(iv) Few candidates gave a complete argument. Many candidates' arguments liberally used ∞ as a number without much success, often wrongly claiming that $y = \infty$ meant that y was bounded. A few gave very clear and concise arguments.

(v) The particular solutions were often hampered by candidates not realising that one of their arbitrary constants had to be zero, but follow through marks were given when appropriate. The expressions for x and y as $t \to \infty$ were sometimes correct, but often were not consistent with the previous solutions, e.g. positive exponential terms disappearing without comment.